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Rarefaction Fans and Dynamic Factoring in Eikonal Equation

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Center for Applied Mathematics
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Isotropic Maze Navigation

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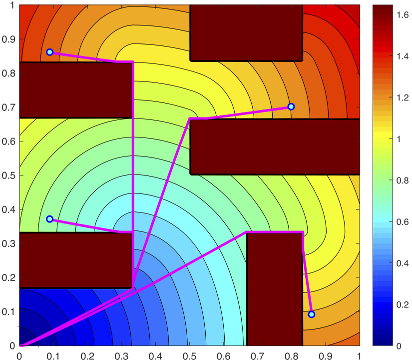
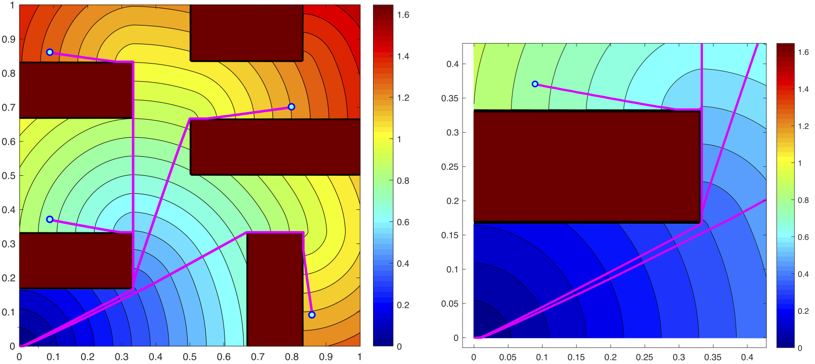


Numerical artifacts at the source and corners.

Isotropic Maze Navigation



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Rarefaction Fans in Eikonal Solution

$$\begin{cases} |\nabla u(\mathbf{x})|F(\mathbf{x}) = 1, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = 0, & \mathbf{x} = \mathbf{x}_0. \end{cases}$$

$$F(x, y) = 0.5x + 0.5.$$

$$u(\mathbf{x}) = 2\text{acosh}\left(1 + 0.25F(\mathbf{x})|\mathbf{x}|\right).\text{[FLZ09, LQ12]}$$

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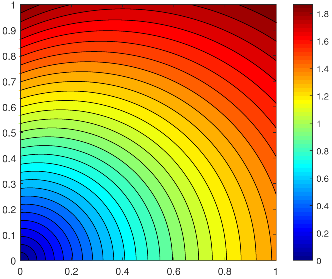

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Singularities in Second-Order Derivatives

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If $F(\mathbf{x}) = 1$ and $\mathbf{x}_0 = (0, 0)$,

$$u(\mathbf{x}) = |\mathbf{x}| = \sqrt{x^2 + y^2}.$$

The second-order derivatives are

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}},$$

$$\frac{\partial^2 u}{\partial x^2} \to \infty, \quad \frac{\partial^2 u}{\partial y^2} \to \infty, \quad x, y \to 0$$

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Outline

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Factored Eikonal Equation

Asymptotic behavior of u:

$$u(\mathbf{x}) \approx T(\mathbf{x}) = \frac{|\mathbf{x} - \tilde{\mathbf{x}}_0|}{F(\tilde{\mathbf{x}}_0)}.$$

$\tilde{\mathbf{x}}_0$ is the source of rarefaction fan.

Additive factoring ($u = T + \tau$) [LQ12]

$$|\nabla T(\mathbf{x}) + \nabla \tau(\mathbf{x})| F(\mathbf{x}) = 1.$$

Multiplicative factoring ($u = T\tau$) [LQ12, FLZ09, TH16]

$$|\nabla T(\mathbf{x})\tau(\mathbf{x}) + T(\mathbf{x})\nabla \tau(\mathbf{x})| F(\mathbf{x}) = 1.$$

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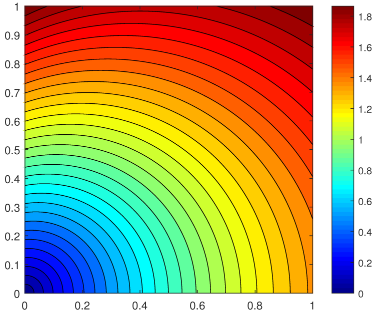
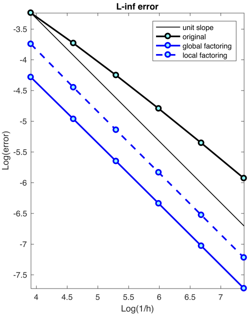
Factored Eikonal Equation

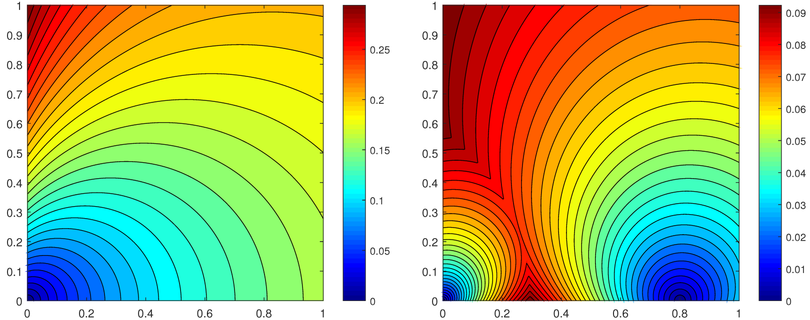


$F(x, y) = 0.5x + 0.5.$



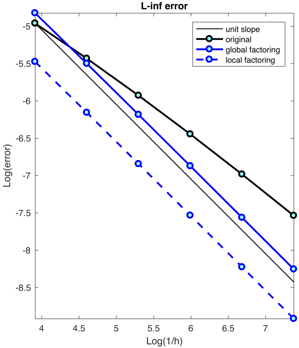
L^∞ error convergence.

Cases Global Factoring is Not The Best

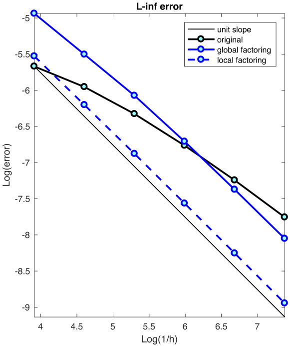


$F(x,y) = 12x + 2.$ $F(x,y) = 5x + 20y + 2.$

Cases Global Factoring is Not The Best



Single source.



Double sources.

Obstacle Induced Rarefaction Fans

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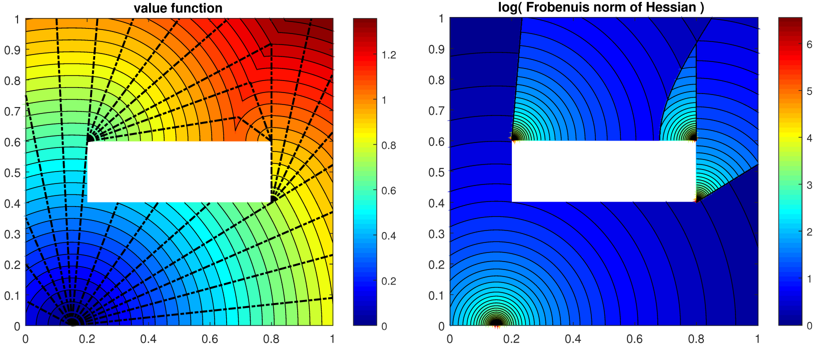
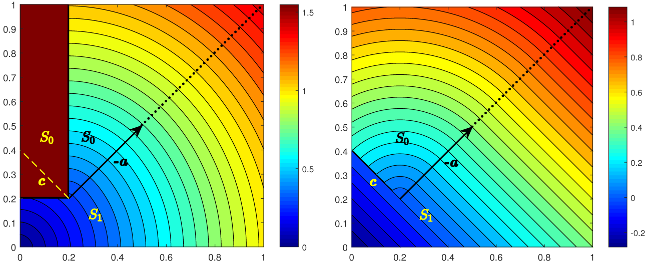


Characteristics. $\log$ of Hessian's norm.

Dynamic Factoring Kernel



$$T(\mathbf{x}) = \begin{cases} \frac{|\mathbf{x}-\tilde{\mathbf{x}}|}{F(\tilde{\mathbf{x}})}, & \mathbf{x} \in S_0 \\ \frac{(-\mathbf{a})\cdot(\mathbf{x}-\tilde{\mathbf{x}})}{F(\tilde{\mathbf{x}})}, & \mathbf{x} \in S_1. \end{cases}$$

Outline

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1. Rarefaction Fans in Eikonal Solution.
2. Dynamic Factoring Techniques.
3. Maze Navigation & Permeable Obstacles.
4. Other Uses.

Maze Navigation

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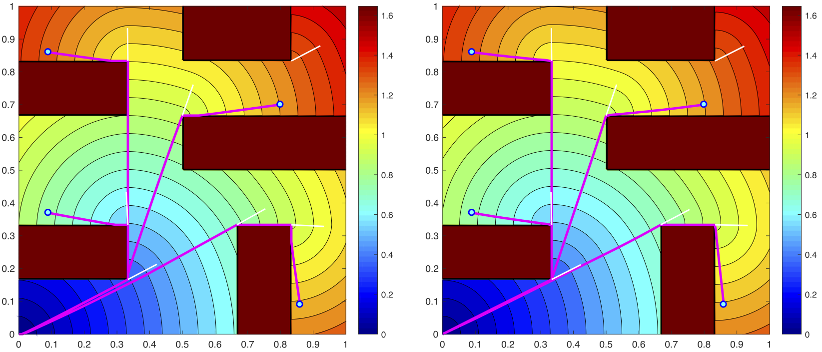

Original FMM. Dynamic factoring.

$$F(x,y) = 1.$$

Maze Navigation

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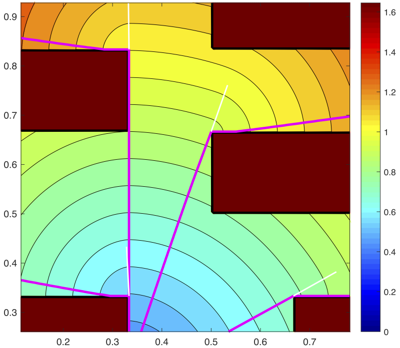
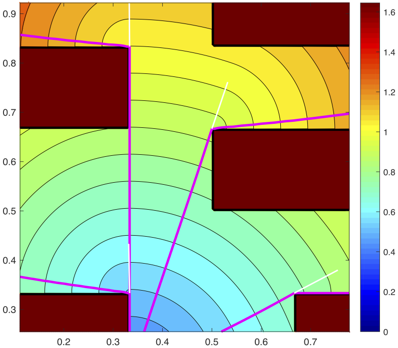


Original FMM.



Dynamic factoring.

$$F(x, y) = 1.$$

Maze Navigation

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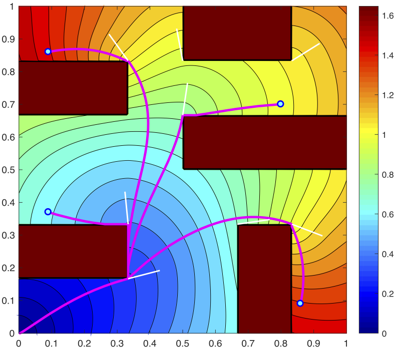

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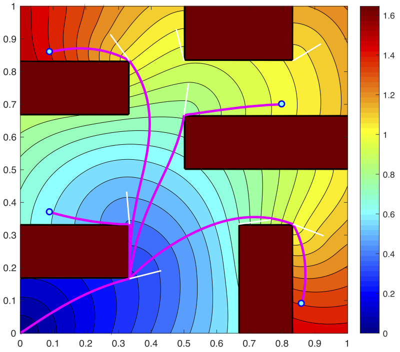
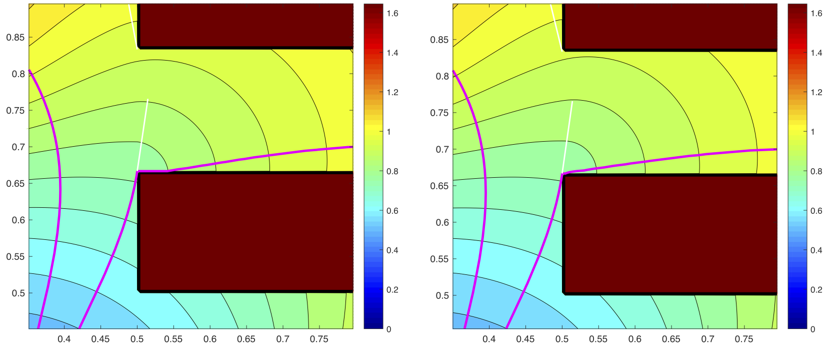


Dynamic factoring.

$$F(x, y) = 1 + 0.5\sin(2\pi x)\sin(2\pi y).$$

Maze Navigation



Original FMM.

Dynamic factoring.

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Maze Navigation: Convergence

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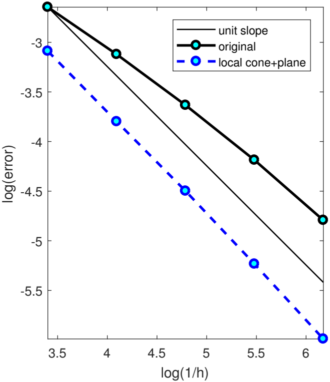
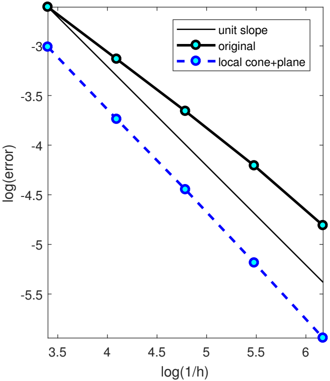
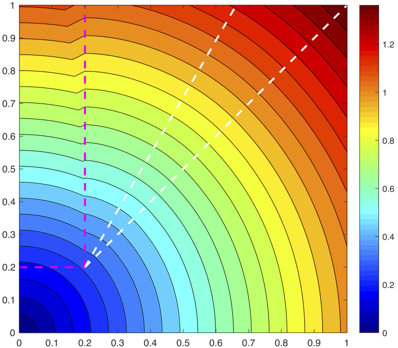


$F(x, y) = 1.$

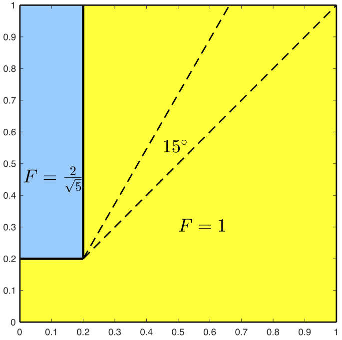


$F(x, y) = 1 + 0.5 \sin(2\pi x) \sin(2\pi y).$

Discontinuous Speed Function

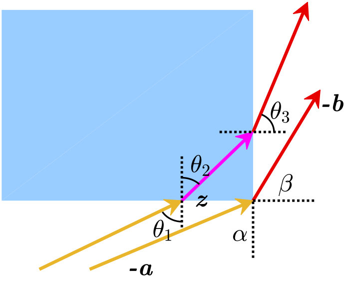


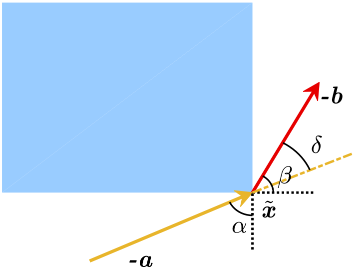
Single permeable obstacle.



Discontinuous speed.

Determine Rarefaction Region by Snell's Law





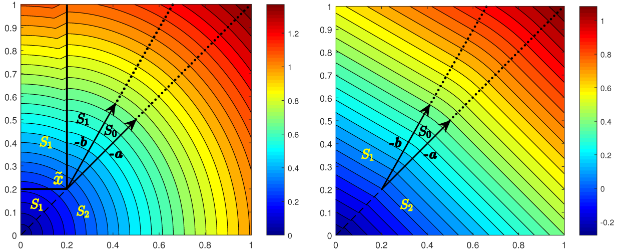
$$\frac{\sin\theta_1}{F_f} = \frac{\sin\theta_2}{F_s},$$

$$\frac{\cos\theta_2}{F_s} = \frac{\sin\theta_3}{F_f}.$$

$$\theta_1 \to \alpha,\ \theta_3 \to \beta,$$

$$\sin\beta = \sqrt{\left(\frac{F_f}{F_s}\right)^2 - \sin^2\alpha}.$$

Factoring Kernel



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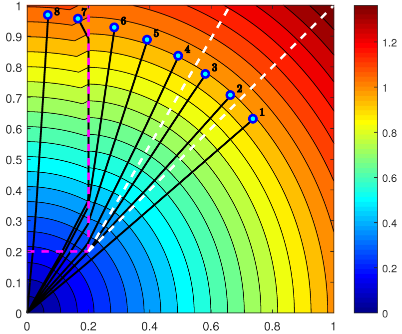
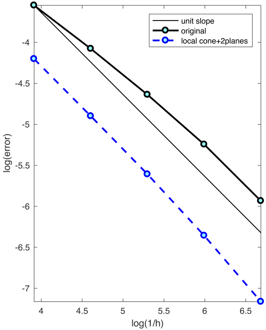
Single Permeable Obstacle



Different trajectories.



L^∞ error convergence.

Several Permeable Obstacles

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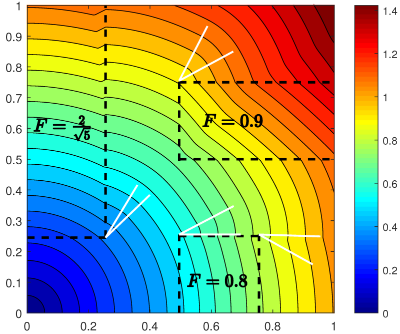
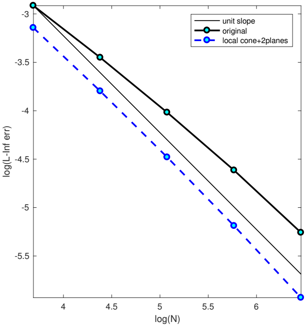


Three permeable obstacles.



L^∞ error convergence.

Outline

Inhomogeneous Permeable Obstacle

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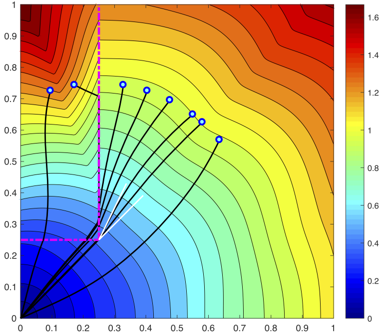
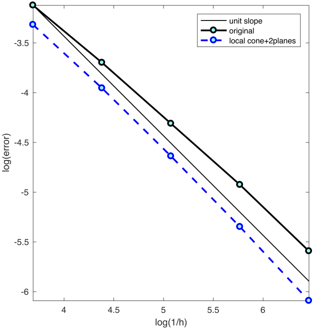
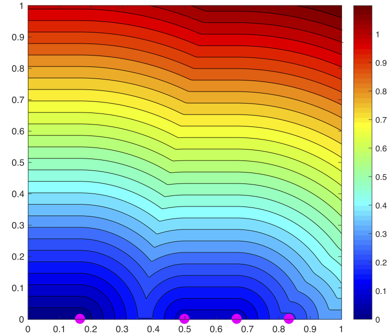


Inhomogeneous permeable obstacle.

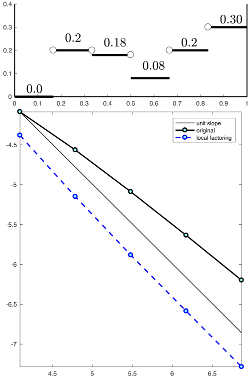


Convergence.

Discontinuous Boundary Condition



Sources of rarefaction fans.



Discontinuous B.C. & Convergence

Conclusion

1. Rarefaction fans may appear due to isolated sources, non-smooth obstacles (boundaries) or discontinuous boundary conditions.
2. Dynamic factoring can discover rarefaction fans automatically and help remove numerical artifacts.

Future works: [QV19]

1. Anisotropic speed function;
2. Polygonal or curved-boundary obstacles;
3. 3D "rarefying edges"?

References I

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Sergey Fomel, Songting Luo, and Hongkai Zhao.
Fast sweeping method for the factored eikonal equation.
Journal of Computational Physics, 228(17):6440–6455, 2009.

Songting Luo and Jianliang Qian.
Fast sweeping methods for factored anisotropic eikonal equations: multiplicative and additive factors.
Journal of Scientific Computing, 52(2):360–382, 2012.

Dongping Qi and Alexander Vladimirsky.
Corner cases, singularities, and dynamic factoring.
Journal of Scientific Computing, 79(3):1456–1476, Jun 2019.

Eran Treister and Eldad Haber.
A fast marching algorithm for the factored eikonal equation.
Journal of Computational Physics, 324:210–225, 2016.

Appendix

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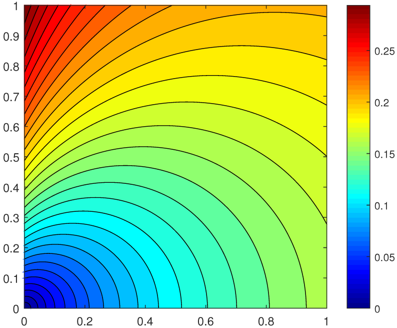
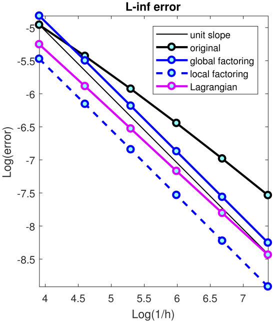
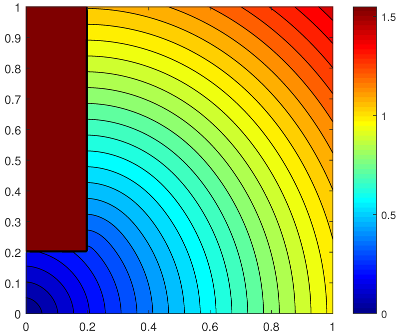


Large gradient.

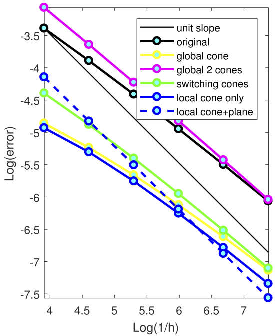


Different methods.

Appendix



Single obstacle.



Different methods.

Appendix

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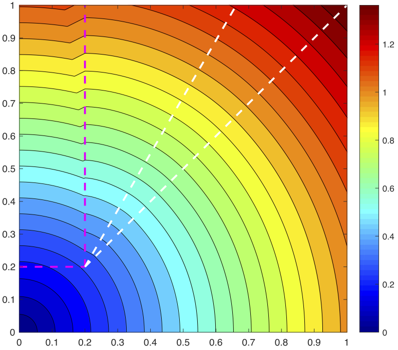
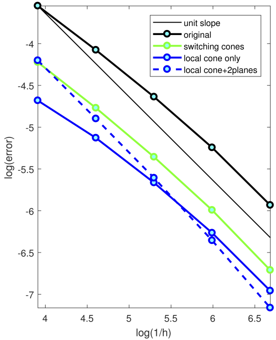


Single permeable obstacle.



Different methods.